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PROGRESS REPORT.

The Managing Committee is g'ad to announce that the Hon'ble Justice Sir Ashutosh Mookerjee, Kt., C.S.I., D.L., D. Sc., Ph. D., Vice-Chancellor of the University of Calcutta, has kindly consented to be enrolled as an Honorary Member of the Society.

The following gentlemen have been admitted as members of the Society :—

1. K. Panchapagesa Iyer, M.A., L.T., Lecturer in Mathematics, Maharajah's College, Pudukottah.

2. G. V. Mahalingam, M.A., L.T., Lecturer in Mathematics, P.R. College, Cocanada.

3. T. V. Sundaresan, B.A., B.E., Assistant Engineer, Development Directorate, Fort, Bombay.

4. S. Ramaswamy Aiyar, M.A., Lecturer in Mathematics, Malabar Christian College, Calicut.

5. S. A. Mani, M.A., L.T., Assistant Lecturer, Government College, Kumbakonam.

6. S. Ramachandra Aiyar, B.A., Lecturer in Mathematics, St. Aloysius' College, Mangalore.

7. R. Krishnamurthy, M.A., Assistant Professor of Mathematics, The Nizam College, Hyderabad.

The Calcutta Mathematical Society has kindly invited our Society to hold the next Conference in Calcutta and accordingly our Fourth Conference will be held in Calcutta in March 1923 during the Easter Holidays. Members are hereby requested to do their utmost to make the Conference a success. The programme will be published in due course. Meanwhile members are requested to prepare and send to the undersigned papers that they may like to read at the Conference, together with short abstracts, before the 31st January 1923. Should the papers be not ready by that time, the abstracts may be sent in advance.

18, PYCROFT'S ROAD,
TRIPLICANE, MADRAS,
25-8-22.

P. V. SESHU AIYAR,
Hon. Joint Secretary.

DETERMINANTS INVOLVING SPECIFIED NUMBERS.

(*Continued from page* 62, *Vol. XIV*, *No.* 2, *J.I.M.S.*)

By C. Krishnamachary & M. Bhimasena Rao.

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§ 1. Let a_n be any function of the integral variable n. Consider the following table :—

0	Σa_1	0	$\Sigma a_1\ \Sigma a_2$	0	$\Sigma a_1\ \Sigma a_2\ \Sigma a_3$
a_1	Σa_2	$a_1\ \Sigma a_2$	$\Sigma a_2\ \Sigma a_3$	$a_1\ \Sigma a_2\ \Sigma a_3$	$\Sigma a_2\ \Sigma a_3\ \Sigma a_4$
a_2	Σa_3	$a_2\ \Sigma a_3$	$\Sigma a_3\ \Sigma a_4$	$a_2\ \Sigma a_3\ \Sigma a_4$	$\Sigma a_3\ \Sigma a_4\ \Sigma a_5$
a_3	Σa_4	$a_3\ \Sigma a_4$	$\Sigma a_4\ \Sigma a_5$	$a_3\ \Sigma a_4\ \Sigma a_5$	
a_4	Σa_5	$a_4\ \Sigma a_5$			

The process of constructing the table is obvious. The second column is obtained from the first by a process of addition. Thus

$$\Sigma a_1 = a_1,\ \Sigma a_2 = a_1 + a_2,\ \ldots$$

The third is obtained from the second by multiplying by the corresponding numbers of the first column. The fourth is obtained from the third by addition, and so on. Thus

$$\Sigma a_1\ \Sigma a_2 = a_1\ \Sigma a_2\ ;$$

$$\Sigma a_2\ \Sigma a_3 = a_2\ \Sigma a_3 + a_1\ \Sigma a_2\ ;$$

Calling a_1, a_2, a_3, the *elements*, and the various *numbers in the even columns*, *the elemental numbers* of the table, we may denote

$$\Sigma a_n\ \Sigma a_{n+1} \ldots \Sigma a_{n+r} \quad \text{by } {}_nA_{n+r}\ ; \tag{1}$$

so that the left suffix indicates the element which commences the Σ and the right suffix the one which ends it. A is employed to denote the numbers defined from the a elements. Similarly, we may use B to denote the elemental numbers defined by the b elements.

The numbers in the *even* columns of the rth row are

$${}_rA_r,\ {}_rA_{r+1},\ {}_rA_{r+2},\ \ldots\ldots$$

The left suffix indicates the row, and the right suffix, the number of the *even* column to which the number belongs ; *e.g.*

$${}_rA_{r+k-1} = \Sigma a_r\ \Sigma a_{r+1} \ldots \Sigma a_{r+k-1}$$

is the number in the kth even column of the rth row.

From the way in which the table is formed, we have

$$\Sigma a_r\ \Sigma a_{r+1} \ldots \Sigma a_m = a_r\ \Sigma a_{r+1}\ \Sigma a_{r+2} \ldots \Sigma a_n + \Sigma a_{r-1}\ \Sigma a_r \ldots \Sigma a_{n-1}{}^{*}$$

Hence the general formula of reduction, *viz.*

$$ {}_rA_n = a_r \cdot {}_{r+1}A_n + {}_{r-1}A_{n-1} \qquad \ldots \qquad (2)$$

1·1. Similarly we may denote

$$\Sigma a_n\ \Sigma a_{n-2} \ldots \Sigma a_{n-2r} \quad \text{by } {}_n\alpha_{n-2r} \qquad \ldots \qquad (3)$$

Here again the left suffix indicates the element which begins the Σ and the right, the one which ends it. Also, we have the general formula of reduction, *viz.*

$$ {}_n\alpha_{n-2r} = a^n \cdot {}_{n-2}\alpha_{n-2r} + {}_{n-1}\alpha_{n-2r-1}. \qquad (3.1)$$

Thus

$$ {}_{2n-1}\alpha_1 = a_{2n-1}\ a_{2n-3} \ldots a_3\ a_1.$$

$$ {}_{2n-2}\alpha_2 = a_{2n-2} \cdot {}_{2n-4}\alpha_2 + a_{2n-3}\ a_{2n-5} \ldots a_3\ a_1. \qquad (3.2)$$

1·2. Let us reduce (2) further for the elemental numbers of the first row as follows,—

$$ {}_1A_n = a_1 \cdot {}_2A_n$$
$$ = a_1 \cdot (a_2 \cdot {}_3A_n + {}_1A_{n-1})$$

i.e.
$$ {}_1A_n - a_1 \cdot {}^1A_{n-1} = a_1\ a_2 \cdot {}^3A_n \qquad \ldots \qquad (4.1)$$
$$ = a_1\ a_2\ (a_3 \cdot {}_4A_n + {}_2A_{n-1}).$$
$$ = a_1\ a_2\ a_3 \cdot {}_4A_n + a_2 \cdot {}_1A_{n-1} \qquad \ldots \qquad (4.2)$$

i.e.
$$ {}_1A_n - (a_1 + a_2)\ {}_1A_{n-1} = a_1\ a_2\ a_3 \cdot {}_4A_n.$$
$$ = a_1\ a_2\ a_3 \cdot (a_4 \cdot {}_5A_n + {}_3A_{n-1})$$
$$ = a_1\ a_2\ a_3\ a_4 \cdot {}_5A_n + a_3\ ({}_1A^{n-1} - a_1 \cdot {}_1A_{n-2})$$

from (4.1).

i.e.
$$ {}_1A_n - (a_1 + a_2 + a_3)\ {}_1A_{n-1} + a_3\ a_1 \cdot {}_1A_{n-2}$$
$$ = a_1\ a_2\ a_3\ a_4 \cdot {}_5A_n \qquad \ldots \qquad \ldots \qquad (4.3)$$
$$ = a_1\ a_2\ a_3\ a_4\ (a_5 \cdot {}_6A_n + {}_4A_{n-1})$$
$$ = a_1\ a_2\ a_3\ a_4\ a_5 \cdot {}_6A_n$$
$$ + a_4\ \{ {}_1A_{n-1} - (a_1 + a_2)\ {}_1A_{n-2} \} \text{ from } (4.2)$$

Rearranging, this may be written

$$ {}_1A_n - {}_4\alpha_4 \cdot {}_1A_{n-1} + {}_4\alpha_2 \cdot {}_1A_{n-2} = a_1\ a_2\ a_3\ a_4\ a_5 \cdot {}_6A_n \ldots \qquad (4.4)$$

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* The use of the Σ is clearly explained in the first section of the last paper. (See Page 55, Vol. *XIV*, No. 2, *J. I. M. S.*) The repetition of the Σ merely stands for repeated summation, the brackets being omitted for convenience.

After r reductions as above, we obtain the following general relation between the α's and the A's.

$$ {}_1A^n - {}^{r-1}\alpha_{r-1}\ {}_1A_{n-1} + {}_{r-1}\alpha_{r-3}\ .\ {}_1A_{n-2} - \ldots $$
$$ + (-1)^{k-1}\ {}_{r-1}\alpha_{r-2k-1}\ .\ {}_1A_{n-k-1} $$
$$ = a_1\ a_2\ a_3\ \ldots\ a_r\ .\ {}_{r+1}A_n, \qquad \ldots \qquad \ldots \quad (4) $$

where $r-2k-1=2$, if $(r-1)$ is even ; and 1 if $(r-1)$ is odd.

The right suffix for α *in the last term is always 2 or 1.* The series on the left is to be continued till then. Remembering the reduction formulæ for α's expressed in (3·1) and (3·2), (4) can be easily proved by induction. (4) is fundamental in the theory of the functions we deal with. It is true whatever be the value of r provided it is less than n. If $r = n$, it is easily seen that (4) reduces to

$$ {}_1A_n - {}_{n-1}\alpha_{n-1}\ .\ {}_1A_{n-1} + {}^{n-1}\alpha_{n-3}\ .\ {}_1A_{n-2} - . = a_1\ a_2\ a_3\ \ldots\ a_n. \quad (4') $$

A question naturally arises, can we find relations so that the last elemental number occurring in (4) is ${}_1A_1$? Or, what is the same thing : what is the value of the left-hand side expression in (4) for values of $r > n$? We can, consistently with the original table, give to r any values which make the right hand suffix of the last A in (4) equal to any positive integer down to unity. We proceed to prove that *if* $r > n$, *the value of the expression on the left is zero.*

1·3. The following purely arithmetical method of establishing the fundamental equations (13) and (14) below is given on account of its directness. The method of § 4 is important in the theory, and is therefore added.

It is obvious from (4) that after $n-1$ reductions (*i.e.* when $r=n-1$), we have

$$ {}_1A_n - {}_{n-2}\alpha_{n-2}\ .\ {}_1A_{n-1} + {}_{n-2}\alpha_{n-4}\ .\ {}_1A_{n-2} - \ldots $$
$$ = a_1 a_2\ \ldots\ a_{n-1}\ .\ {}_nA_n \qquad \ldots \qquad \ldots \qquad \ldots \quad (4.5) $$

the series on the left being continued till the right suffix of α *is 1 or 2.*

For the sake of clearness, let $n = 2m$ and omit the left suffix of A's. The above equation is

$$ A_{2m} - {}_{2m-2}\alpha_{2m-2}\ .\ A_{2m-1} + {}_{2m-2}\alpha_{2m-4}\ .\ A_{2m-2} - \ldots $$
$$ + (-1)^r\ {}_{2m-2}\alpha_{2m-2r}\ .\ A_{2m-r} + \ldots + (-1)^{m-1}\ {}_{2m-2}\alpha_2\ .\ A_{m+1} $$
$$ = a_1\ a_2\ \ldots\ a_{2m-1}\ .\ {}_{2m}A_{2m} \qquad \ldots \qquad \ldots \quad (4.6) $$

Let $n = 2m-1$. Then,

$$A_{2m-1} - {}_{2m-3}\alpha_{2m-3} \cdot A_{2m-2} + {}_{2m-3}\alpha_{2m-5} \cdot A_{2m-3} - \ldots$$
$$+ (-1)^r \, {}_{2m-3}\alpha_{2m-2r-1} \cdot A_{2m-2r-1} + \ldots + (-1)^{m-1} \, {}_{2m-3}\alpha_1 . A_m$$
$$= a_1 \, a_2 \ldots a_{2m-2} \cdot {}_{2m-1}A_{2m-1} \cdot \ldots \qquad (4{\cdot}7)$$

Now from (4.6),

$$A_{2m} - {}_{2m-2}\alpha_{2m-2} \cdot A_{2m-1} + {}_{2m-2}\alpha_{2m-4} \cdot A_{2m-2} - \ldots$$
$$+ (-1)^r \, {}_{2m-2}\alpha_{2m-2r} \cdot A_{2m-r} + \ldots + (-1)^{m-1} \, {}_{2m-2}\alpha_2 . A_{m+1}$$
$$= a_1 \, a_2 \ldots a_{2m-1} (a_{2m} + {}_{2m-1}A_{2m-1})$$
$$= a_1 \, a_2 \ldots a_{2m} + a_{2m-1} \{ A_{2m-1} - {}_{2m-3}\alpha_{2m-3} \cdot A_{2m-2} + \ldots$$
$$+ (-1)^r \, {}_{2m-3}\alpha_{2m-2r-1} \cdot A_{2m-r-1} + \ldots + (-1)^{m-1} \, {}_{2m-3}\alpha_1 \cdot A_m \}$$

by substituting from (4·7). Transposing the terms within the flower brackets on the right, to the left and remembering (3·1), we obtain,

$$A_{2m} - {}_{2m-1}\alpha_{2m-1} \cdot A_{2m-1} + {}_{2m-1}\alpha_{2m-3} \cdot A_{2m-2} - \ldots + (-1)^r$$
$${}_{2m-1}\alpha_{2m-3r+1} \cdot A_{2m-r} + \ldots + (-1)^m {}_{2m-1}\alpha_1 \cdot A_m = a_1 \, a_2 \ldots a_{2m}, \quad (4{\cdot}8)$$

which is the formula (4′) for $n=2m$. Similarly we can obtain (4′) when $n=2m-1$, *viz.*

$$A_{2m-1} - {}_{2m-2}\alpha_{2m-2} \cdot A_{2m-2} + \ldots + (-1)^r \, {}_{2n-2}\alpha_{2m-2r} \cdot A_{2m-r-1}$$
$$+ (-1)_{m-1} \, {}_{2m-2}\alpha_2 \cdot A_m = a_1 a_2 . \; a_{2m-1} . \quad (4{\cdot}9)$$

Substituting from (4·9) in (4·8) for $a_1 \, a_2 \ldots a_{2m-1}$, we have,

$$A_{2m} - {}_{2m-1}\alpha_{2m-1} \cdot A_{2m-1} + {}_{2m-1}\alpha_{2m-3} \cdot A_{2m-2} - \ldots$$
$$+ (-1)^r {}_{2m-1}\alpha_{2m-2r+1} \cdot A_{2m-r} + \ldots + (-1)^m {}_{2m-1}\alpha_1 \cdot A_m$$
$$= a_{2m} \{ A_{2m-1} - {}_{2n-2}\alpha_{2m-2} \cdot A_{2m-2} + \ldots$$
$$+ (-1)^r {}_{2m-2}\alpha_{2m-2r} A_{2m-r-1} + \ldots + (-1)^{n-1} \, {}_{2m-2}\alpha_2 \cdot A_m \}.$$

i.e.

$$A_{2m} - {}_{2m}\alpha_{2m} \cdot A_{2m-1} + {}_{2m}\alpha_{2m-2} . A_{2m-2} - \ldots$$
$$(-1)^r {}_{2m}\alpha_{2m-2r+2} . A_{2m-r} + \ldots + (-1)^m \, {}_{2m}\alpha_2 \; A_m \} = 0. \qquad (4{\cdot}10)$$

Similarly from (4·9), by substituting for $a_1 a_2 \ldots a_{2m-2}$,

$$A_{2m-1} - {}_{2m-1}\alpha_{2m-1} \cdot A_{2m-2} + {}_{2m-1}\alpha_{2m-3} \cdot A_{2m-3} - \ldots$$
$$+ -(1)^r \, {}_{2m-1}\alpha_{2m-2r+1} \cdot A_{2m-r-1} + \ldots + (-1)^m {}_{2m-1}\alpha_1 \cdot A_{m-1} = 0.$$
$$(4{\cdot}11)$$

From (4·10), writing $m-1$ for m, we have

$$A_{2m-2} - {}_{2m-2}a_{2m-2}\,A_{2m-3} + \ldots + (-1)^r {}_{2m-2}a_{2m-2r}\cdot A_{2m-r-2}$$
$$+\ldots+(-1)^{m-1}{}_{2m-2}a_2\cdot A_{m-1}=0. \qquad (4\cdot12)$$

Multiply (4·12) by $-a_{2m}$, and add to (4·11). We get

$$A_{2m-1} - {}_{2m}a_{2m}\cdot A_{2m-2} + {}_{2m}a_{2m-2}\cdot A_{2m-3} - \ldots$$
$$+(-1)^m {}_{2m}a_2\cdot \; A_{m-1}=0. \qquad (4\cdot13)$$

Similarly to (4·12) add $-a_{2m-1}$ times the expression on the left in (4·11) after writing $m-1$ for m. We obtain,

$$A_{2m-2} - {}_{2m-1}a_{2m-1}\,A_{2m-3} + \ldots$$
$$+(-1)^m {}_{2m-1}a_1\,A_{m-2}=0 \quad (4\cdot14)$$

The equations (4·10) and (4·13) are the first two of the equations in (14·3) § 4, below; and (4·11) and (4·14) are the last two of the equations in (13·3). It is now obvious that the other equations in (13·3) and (14·3) can be similarly obtained *in a purely arithmetical manner* without any reference to the method in § 4.

§ 2. Let

$$M(x) = \frac{x^r}{r!} - \Sigma a_{r+1}\frac{x^{r+2}}{r+2} + \Sigma a_{r+1}\Sigma a_{r+2}\frac{x^{r+4}}{r+4!} - \ldots$$
$$= \frac{x^r}{r!} - {}_{r+1}A_{r+1}\frac{x^{r+2}}{r+2} + {}_{r+1}A_{r+2}\frac{x^{r+4}}{r+4!} - \ldots \qquad (5)$$

the coefficients A being the elemental numbers in the even columns of the $(r+1)^{\text{th}}$ row. Thus,

$$M_0(x) = 1 - \Sigma a_1\cdot\frac{x^2}{2!} + \Sigma a_1\Sigma a_2\cdot\frac{x^4}{4!} - \ldots$$
$$= 1 - {}_1A_1\frac{x^2}{2!} + {}_1A_2\cdot\frac{x^4}{4!} - \ldots$$
$$M_1(x) = \frac{x}{1!} - {}_2A_2\frac{x^3}{3!} + {}_2A_3\cdot\frac{x^5}{5!} - \ldots$$

The functions $M^r(x)$ are very interesting and general, and because of their wide generality, we propose to deal with their properties at some length in a future paper. In earlier papers *, we have identified the functions $M_r(x)$ with well known functions as follows.—

(1) $a_n = n^2$. $\quad {}_1A = {}_nE_n$, n^{th} Eulerian number.

$$M_0(x) = \text{sech}\, x,\ M_1(x) = \text{sech}\, x \tanh x$$
$$M_r(x) = \frac{\text{sech}\ x\ (\tanh x)^r}{r!}. \qquad (6)$$

* "Some properties of Eulerian and prepared Bernoullian numbers" presented to the Third Conference of the Indian Mathematical Society.

(2) $a = r\ (n+r-1)$.

$$M_0\ (x) = (\text{sech}\ x)^n, \qquad M_1\ (x) = (\text{sech}\ x)^n \tanh x.$$

$$M_r\ (x) = (\text{sech}\ x)^n \frac{(\tanh x)^r}{r\,!}. \tag{7}$$

(3) $a_n = n^*$. $\quad {}_1A_n = 1 \cdot 3 \cdot 5 \ldots (2n-1)$.

$$M_0\ (x) = e^{-x^2/2}, \quad M_r\ (x) = \frac{x_r}{r\,!} \cdot e^{-x^2/2}. \tag{8}$$

$$M_0\ (x) = (\text{sech}\ x\sqrt{a})^{\frac{b}{a}+1}.$$

(4) $\quad a_n = an + bn. \qquad M^r\ (x) = (\text{sech}\ x\sqrt{a})^{\frac{a}{b}+1} \dfrac{(\tanh x\sqrt{a})^r}{r\,!\ \ a^{\frac{1}{2}r}}.$

(5) $\quad a_n = 1.\ M_0\ (x) = \dfrac{1}{x} J_1\ (2x)$

$$M_r\ (x) = \frac{r+1}{x} J_{r+1}\ (2x) = J_r\ (2x) + J_{r+2}\ (2\,x).$$

In fact

$$M_r\ (x) = \frac{x^r}{r\,!} - \frac{r+1}{1\,!} \cdot \frac{x^{r+2}}{r+2\,!} + \frac{(r+1)\ (r+4)}{2\,!}\ \frac{x^{r+4}}{r+4\,!} - \frac{(r+1)\ (r+5)}{3\,!}\ \frac{(r+6)}{}\ \frac{x^{r+6}}{r+6\,!} + \ldots$$

[A table of values is found in Appendix I.]

The following algebraical method of proof may be found interesting :

$$\Sigma a_{r+1} = 1+1+\ldots \text{ to } r+1 \text{ terms } = \frac{r+1}{1}.$$

$\Sigma a_{r+1}\ \Sigma a_{r+2}$ = sum of $r+1$ terms of the series Σa_{r+2}

$$= \frac{r+2}{1} + \frac{r+1}{1} + \ldots + \frac{2}{1}. = \frac{(r+1)\ (r+4)}{1 \cdot 2}$$

---

* The cas of $a_n = n$ presents remarkable simplicity in the evaluation of $a_n\ \Sigma a_{n+1} \ldots \Sigma a_{n+r}$. Thus

$$\Sigma a_{n+1} = \frac{(n+1)\ (n+2)}{2}$$

$$\Sigma a_n\ \Sigma a_{n+1} = \sum_1^n \frac{n\ (n+1)\ (n+2)}{2}.$$

$$= \frac{n\ (n+1)\ (n+2)\ (n+3)}{2 \cdot 4},$$

$\Sigma a_{r+1} \Sigma a_{r+2} \Sigma a_{r+3}$ = sum of $(r+1)$ terms of the series whose last term is $\Sigma a_{r+2} \Sigma a_{r+}$

= sum of $(r+1)$ terms of the series whose rth term is

$$\frac{(r+1)\ (r+4}{1\cdot2}$$

$$= \frac{(r+1)\ (r+5)\ (r+6)}{1\cdot2\cdot3} \quad \text{and so on.}$$

§ 3. Since ${}_{1}A_n = a_1 \cdot {}_{2}A_n$, we have

$$\frac{d}{dx} M_0 (x) = -a_1 M_1 (x). \qquad (11\cdot1)$$

In virtue of the recurrence formula (2), we have the general and fundamental relation between any three consecutive series,

$$\frac{d}{dx} M_r (x) = M_{r-1} (x) - a_{r+1} M_{r+1} (x). \qquad (11)$$

This relation is true for all positive integral values of r and for $r=0$, if we consider $M_{-1} (x) = 0$ in virtue of (11·1).

From (11), it follows that $M_r (x)$ can be expressed in terms of $M_0 (x)$ and its differential coefficients.

And we easily obtain

$$(D^r + {}_{r-1}a_{r-1} D^{r-2} + {}_{r-1}a_{r-3} D^{r-4} + \ldots) M_0 (x)$$
$$= (-1)^r a_1 a_2 \ldots a_r M_r (x). \qquad (12)$$

This is directly proved by induction. The last term on the left is

$${}_{r-1}a_1 = a_{r-1} a_{r-3} \ldots a_3 a_1, \text{ if } r \text{ is even};$$

and $${}_{r-1}a_2 \cdot D = \Sigma a_{r-1} \Sigma a_{r-3} \ldots \Sigma a_2 D, \text{ if } r \text{ is odd.} \qquad (12\cdot3)$$

Following Brioschi (Muir, *Theory of Determinants*, Vol. II, page 344), we can write the equation (12) in the form

$$\begin{vmatrix} D & a_1 & \ldots & \ldots \\ -1 & D & a_2 & \ldots \\ \cdot & -1 & D & a_3 & \ldots \end{vmatrix} M_0 (x) = (-1)^r a_1 a_2 \ldots a_r M_r (x). \qquad (12\cdot4)$$

there being r rows and columns, D standing for $\frac{d}{dx}$.

It will be proved in a continuation of this paper that the denominators of the convergents of the continued fraction

$$\frac{1}{D-}\ \frac{a_1}{D-}\ \frac{a_2}{D-}\ \cdots \tag{12·5}$$

is the expression on the left hand side in (12).

3·1. One interesting point about (12) may be noticed in passing, viz. whenever one of the elements a_r vanishes, the right hand side in (12) vanishes, and hence we can obtain the function $M_0(x)$ by solving a linear differential equation with constant co-efficients. Since such an equation can always be solved (at least theoretically), the function $M_0(x)$ can in such a case be determined completely, but for constants.

Ex. 1. Write $a_n = n(n-3)$. $a_3 = 0$.

The equation (12) is

$$\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 0.$$

$$\therefore \qquad y = A + B\cosh 2x + c\sinh 2x.$$

Here $a_n = n^2 - [3n$, the case in § 2, (4) where $a = 1, b = -3,]$ so that $M_0(x) = \cosh^2 x = \frac{1}{2}(1 + \cosh 2x)$.

Since $$\frac{d}{dx} M_0(x) = -a_1 M_1(x) = 2M_1(x),$$

the above equation can be written

$$\left(\frac{d^2}{dx^2} - 4\right) M_1(x) = 0.$$

i.e. $$M_1(x) = A\cosh 2x + B\sinh 2x.$$

From § 2, (4), $M_1(x) = \cosh^2 x \tanh x$.
$= \frac{1}{2}\sinh 2x$.

Ex. 2. Write $a_n = (n-2)(n-3)$.

The differential equation for $M_0(x)$ is

$$\frac{d^2y}{dx^2} + 2y = 0.$$

$$\therefore \quad M_0(x) = A\cos(\sqrt{2}x) + B\sin(\sqrt{2}x).$$

By forming the table, it will be seen that the first row gives the co-efficients in $\cos(\sqrt{2}x)$ viz. $2, 2^2, 2^3, \ldots\ldots$ Similarly but for a factor $\sqrt{2}$, the co-efficients in the second row will be found to be the co-efficients of $\sin\sqrt{2}x$.

3·2. Another advantage of (12) is that $M_r(x)$ can be found in all cases provided $M_0(x)$ is known.

Ex. $a_n = n^2$. Here (12) gives, for $r = 3$,

$$\frac{d^3y}{dx^3} + 5\frac{dy}{dx} = (D^3 + 5D)\ \mathrm{sech}\ x.$$

$$= -1^2 \cdot 2^2 \cdot 3^2 \cdot \left(\frac{\mathrm{sech}\ x \tanh^3 x}{3!}\right). \text{ [Cf. (6) above.]}$$

§ 4. In (12), write $r = 2n$, and equate the co-efficients of x^{2s} on both sides. We obtain the following equations.

If $s > m$,

$$A_{m+s} - {}_{2m-1}\alpha_{2m-1} \cdot A_{m+s-1} + {}_{2m-1}\alpha_{2m-3} \cdot A_{m+s-2} - \ldots + (-1)^m\ {}_{2m-1}\alpha_1 \cdot A_s = a_1 a_2 \ldots a_{2m} \cdot {}_{2m+1}A_{m+s}. \quad (13 \cdot 1)$$

If $s = m$,

$$A_{2m} - {}_{2m-1}\alpha_{2m-1} A_{2m-1} + {}_{2m-1}\alpha_{2m-3} A_{2m-2} - \ldots + (-1)^m\ {}_{2m-1}\alpha_1 \cdot A_m = a_1 a_2 \ldots a_{2m}. \quad (13 \cdot 2)$$

If $s < m$,

$$\left.\begin{aligned}
&A_m - {}_{2m-1}\alpha_{2m-1} \cdot A_{m-1} + {}_{2m-1}\alpha_{2m-3} \cdot A_{m-2} - \ldots + (-1)^m\ {}_{2m-1}\alpha_1 \cdot A_0 = 0.\\
&A_{m+1} - {}_{2m-1}\alpha_{2m-1} A_m + {}_{2m-1}\alpha_{2m-3} A_{m-1} - \ldots + (-1)^m\ {}_{2m-1}\alpha_1 \cdot A_1 = 0.\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
&A_{2m-1} - {}_{2m-1}\alpha_{2m-1} \cdot A_{2m-2} + {}_{2m-1}\alpha_{2m-3} \cdot A_{2m-3} - \ldots + (-1)^m\ {}_{2m-1}\alpha_1 \cdot A_{m-1} = 0.
\end{aligned}\right\} \ldots (13 \cdot 3)$$

where $A_0 = 1$.

4·1. Similarly in the differential equation, write $r = 2m + 1$, and equate the co-efficients of x^{2s+1} on both sides. We obtain the following equations :

If $s > m$,

$$A_{m+s+1} - {}_{2m}a_{2m} \cdot A_{m+s} + {}_{2m}a_{2m-2} \cdot A_{m+s-1} - \dots + (-1)^m {}_{2m}a_2 \, A_{s+1} = a_1 a_2 \dots a_{2m+1} \cdot {}_{2m+2}A_{s+m+1}. \quad (14\cdot1)$$

If $s = m$,

$$A_{2m+1} - {}_{2m}a_{2m} \, A_{2m} + {}_{2m}a_{2m-2} \, A_{2m-1} - \dots + (-)^m {}_{2m}a_2 \, A_{m+1} = a_1 a_2 \dots a_{2m+1}. \quad (14\cdot2)$$

If $s < m$,

$$\left.\begin{aligned} A_{2m} - {}_{2m}a_{2m} \, A_{2m-1} + {}_{2m}a_{2m-2} \, A_{2m-2} - \dots + (-1)^m \, {}_{2m}a_2 \, A_m = 0. \\ A_{2m-1} - {}_{2m}a_{2m} \cdot A_{2m-2} + {}_{2m}a_{2m-2} \cdot A_{2m-3} - \dots + (-1)^m \cdot {}_{2m}a_2 \cdot A_{m-1} = 0. \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ A_{m+1} - {}_{2m}a_{2m} \cdot A_m + {}_{2m}a_{2m-2} \, A_{m-1} - \dots + (-1)^m \, {}_{2m}a_2 \cdot A_1 = 0. \end{aligned}\right\} \dots \quad (14\cdot3)$$

§ 5. From the equations (13...) and (14...), we may evaluate determinants whose elements are the first row elemental numbers of the table. The method of procedure is exactly similar to the one we adopted in our last paper. We content ourselves with stating the main results. We add some examples of cases in which the elements are different from those already considered.

$$\begin{vmatrix} A_0 & A_1 & A_2 & \dots & A_m \\ A_1 & A_2 & A_3 & \dots & A_{m+1} \\ \dots & \dots & \dots & \dots & \dots \\ A_m & A_{m+1} & A_{m+2} & \dots & A_{2m} \end{vmatrix} = (a_1 a_2)^m (a_3 a_4)^{m-1} (a_5 a_6)^{m-2} \dots (a_{2m-1} a_{2m})^1 \quad (15)$$

$$\begin{vmatrix} A_1 & A_2 & A_3 & \dots & A_m \\ A_2 & A_3 & A_4 & \dots & A_{m+1} \\ \dots & \dots & \dots & \dots & \dots \\ A_m & A_{m+1} & A_{m+2} & \dots & A_{2m-1} \end{vmatrix} = a_1^m (a_2 a_3)^{m-1} (a_4 a_5)^{m-2} \dots (a_{2m-2} a_{2m-1})^1 \quad (16)$$

$$\begin{vmatrix} A_2 & A_3 & \dots & A_{m+1} \\ A_3 & A_4 & \dots & A_{m+2} \\ \dots & \dots & \dots & \dots \\ A_{m+1} & A_{m+2} & \dots & A_{2m} \end{vmatrix} = a_1^m (a_2 a_3)^{m-1} (a_4 a_5)^{m-2} \dots (a_{2m-2} a_{2m-1}) \times {}_{2m}a_2. \quad (17)$$

$$\begin{vmatrix} A_0 & A_1 & A_2 & \dots & A_m \\ A_1 & A_2 & A_3 & \dots & A_{m+1} \\ \dots & \dots & \dots & \dots & \dots \\ A_{m-2} & A_{m-1} & \dots & \dots & A_{2m-2} \\ A_{m-1} & A_m & \dots & \dots & A_{2m-1} \\ A_s & A_{s+1} & \dots & \dots & A_{m+s} \end{vmatrix} = (a_1 a_2)^m (a_3 a_4)^{m-1} \dots (a_{2m-1} a_{2m})' \times {}_{2m+1}A_{s+m}. \qquad (18)$$

where s is any integer equal to, or, greater than m, ${}_{2m+1}A_{2m}$ being equal to unity.

$$\begin{vmatrix} A_1 & A_2 & A_3 & \dots & A_{m+1} \\ A_2 & A_3 & A_4 & \dots & A_{m+2} \\ \dots & \dots & \dots & \dots & \dots \\ A_{m-1} & A_m & \dots & \dots & A_{2m-1} \\ A_m & A_{m+1} & \dots & \dots & A_{2m} \\ A_{s+1} & A_{s+2} & \dots & \dots & A_{m+s+1} \end{vmatrix} = a_1^{m+1} (a_2 a_3)^m \dots (a_{2m-2} a_{2m-1})^2 (a_{2m} a_{2m+1})^1 \times {}_{2m+2}A_{m+s+1}. \qquad (19)$$

where s is any integer equal to, or greater than m.

5.1. Particular cases of the above determinants were obtained independently in the last paper, in view of the fact that their elements are Euler's and Bernoulli's numbers.

We give here other examples.

(a) $a_n = n$. $\qquad {}_1A_n = 1.3.5 \dots (2n-1)$

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & 15 \\ 3 & 15 & 105 \end{vmatrix} = 48 = (1.2)^2 (3.4)^1 = (1.2)^2 (3.4)^1 \qquad \begin{vmatrix} 1 & 3 & 15 \\ 3 & 15 & 105 \\ 15 & 105 & 945 \end{vmatrix} = 720 = 1(2.3)(4.5)^2 = 1^3 (2.3)^2 (4.5).$$

(b) $a_1 = a_2 = \dots = a_n = 1$.

The values of the A's are calculated in Appendix I with the help of which we write down some examples.

$$\begin{vmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \end{vmatrix} = 1, \qquad \begin{vmatrix} 2 & 5 & 14 \\ 5 & 14 & 42 \\ 14 & 42 & 132 \end{vmatrix} = 4 \text{ and } {}_6\alpha_2 = 4.$$

Again $\qquad {}_n\alpha_n = \Sigma a_n = n.$

$${}_n\alpha_{n-2} = (n-2) + (n-3) + \dots + 1$$

$$= \frac{(n-2)(n-1)}{2!}.$$

$$n^{a}{}_{n-4} = \text{sum of terms } (\Sigma a_{n-2}\ \Sigma a_{n-4}),\ (n-4) \text{ in number}$$

$$= \frac{1}{2!}\{1\cdot2 + 2\cdot3 + \ldots + (n-4)(n-3)\}$$

$$= \frac{1}{3!}(n-4)(n-3)(n-2),$$

$$n^{a}{}_{n-6} = \text{sum of terms } (\Sigma a_{n-2}\ \Sigma a_{n-4}\ \Sigma a_{n-6}),\ (n-6) \text{ in number.}$$

$$= \frac{1}{3!}\{1.2.3 + 2.3.4 + 3.4.5 + \ldots\}$$

$$= \frac{1}{4!}(n-6)(n-5)(n-4)(n-3);$$

and generally $n^{a}{}_{n-2r} = \dfrac{1}{r+1!}\ (n-2r)(n-2r+1)\ldots(n-r-1)(n-r).$

§ 6. We give further examples of important coefficients being obtained from the table. Their proof depends upon a fundamental result relating to the representation by a continued fraction of the integral

$$\int_0^\infty M^\circ(x)\, e^{-xt}\ dx.$$

That result being established, the examples given here follow easily from the continued fractions given by Prof. L. J. Rogers in his paper, "*Asymptotic Series as Convergent Continued Fractions*" in the Proceedings of the London Mathematical Society, Series II, Vol. IV.

(*a*) Let $a_{2n-1} = (2n-1)^2\ k^2,\ a_{2n} = (2n)^2.$

The first row numbers of the table give the coefficients in the expansion in an ascending series of powers of x of the function dn (x, k).

i.e. $\qquad M_0(x) = \text{dn}(x, k).$

$$M_{2r}(x) = \frac{\text{sn}^{2r}(x,k)\,\text{dn}(x,k)}{2r!},\quad M_{2r+1}(x) = \frac{\text{sn}^{2r+1}(x,k)\,\text{cn}(x,k)}{2r+1!}.$$

(*b*) Let $a_{2n-1} = (2n-1)^2,\ a_{2n} = (2n)^2\ k^2.$

$$M_0(x) = \text{cn}(x, k).$$

$$M_{2r}(x) = \frac{\text{sn}^{2r}(x,k)\,\text{cn}(x,k)}{2r!}.\quad M_{2r+1}(x) = \frac{\text{sn}^{2r+1}(x,k)\,\text{dn}(x,k)}{2r+1!}$$

(c) Let $a_{2n-1} = a_{2n} = n;$

$${}_1A_n = n!,\ {}_2A_n = n+1!,\ {}_3A_n = (n+1)(n+1)!\ \text{ etc.}$$

(*d*) Let $a_1 = m$, $a_2 = 1$, $a_3 = m+1$, $a_4 = 2, \ldots\ldots$

$a_{2n-1} = m + n_{m+n-1}$, $a_{2n} = n, \ldots$

$${}_1A_n = m\,(m+1)\,(m+2)\,\ldots\,(m+n-1)$$

(*e*) Let $a_n = \dfrac{n^4}{(2n-1)(2n+1)}$.

$${}_1A_n = 2\,(2^{2n-1} - 1)\,B_n,$$

where B_n is the nth Bernoullian number. We may write down the values of some determinants here. If $c_n = (2^{2n-1} - 1)\,B_n$, we have

$$\begin{vmatrix} 1 & c_1 & \ldots & \ldots & c_n \\ c_1 & c_2 & \ldots & \ldots & c_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_n & c_{n+1} & \ldots & \ldots & c_{2n} \end{vmatrix} = \frac{1}{2^{n+1}} \cdot \frac{[(1.2)^n\,(3.4)^{n-1} \ldots (2n-1, 2n)^1]^4}{3^{2n}\,5^{2n-1}\,7^{2n-2} \ldots (4n-1)^2\,(4n+1)^1}$$

$$\begin{vmatrix} c_1 & c_2 & \ldots & c_n \\ c_2 & c_3 & \ldots & c_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_n & c_{n+1} & \ldots & c_{2n-1} \end{vmatrix} = \frac{1}{2^n} \cdot \frac{[1_n (2\cdot3)^{n-1} (4\cdot5)^{n-2} \ldots (2n-2.2n-1)^1]^4}{3^{2n-1}\,5^{2n-2} \ldots (4n-3)^2\,(4n-1)^1}$$

(*f*) Let $a_1 = \dfrac{1.2^2.3}{2^2.3.5}$, $a_2 = \dfrac{2.3^2\cdot4}{2^2.5.7}$, $a_n = \dfrac{n(n+1)^2\,(n+2)}{2^2\,(2n+1)\,(2n+3)}$.

${}_1A_n = 6\,B_{n+1}$, where B_n is the nth Bernoullian number. The construction of a table in (*e*) or (*f*) is out of the question owing to the obvious tediousness of the work. But it is given here in connection with the evaluation of determinants whose elements are Bernoullian numbers, or involve them. Hence,

$$\begin{vmatrix} B_1 & B_2 & \ldots & B_{n+1} \\ B_2 & B_3 & \ldots & B_{n+2} \\ \ldots & \ldots & \ldots & \ldots \\ B_{n+1} & & \ldots & B_{2n+1} \end{vmatrix} = \frac{1}{2^{n+1}} \cdot 1^{2n} \cdot 2^{2n-1}\,2^{2n-2} \cdot 3^{2n-3}.\; 3^{2n-4} \ldots n^3 \cdot n^2\,(n+1) \times \frac{3^{2n} \cdot 3^{2n-1} \cdot 5^{2n-2}\,5^{2n-3} \ldots (2n+1)^2 (2n+1)}{3^{2n+1}\,5^{2n}\,7^{2n-1} \ldots (4n+1)^2\,(4n+3)}.$$

$$\begin{vmatrix} B_2 & B_3 & \ldots & B_{n+1} \\ B_2 & B_4 & \ldots & B_{n+2} \\ \ldots & \ldots & \ldots & \ldots \\ B_{n+1} & B_{n+2} & \ldots & \ldots B_{2n} \end{vmatrix} = \frac{1}{6^n}\,(1\cdot2^3)^{n-1}\,(2\cdot3^3)^{n-2}\,(3\cdot4^3)^{n-3}. \ldots (\overline{n-1}\,n^3)^1. \times \frac{(3^3\cdot5)^{n-1}\,(5^3\cdot7)^{n-2}\,(7^3\cdot9)^{n-3} \ldots \times (\overline{2n-1}^3.2n+1)}{5^{2n-1}\,7^{2n-2}\,9^{2n-3} \ldots (4n-1)^2\,(4n+1)}.$$

(g) Let $a_1 = \frac{1^2}{1\cdot 3}, a_2 = \frac{2^2}{3\cdot 5}, \ldots a_n = \frac{n^2}{(2n-1)\,(2n+1)}$.

Then $$ {}_1A_n = \frac{1}{2n+1}. $$

We therefore evaluate the following two determinants given by Rouche (1858) in another connection. It does not appear however that the determinants in question were evaluated by him. (See Muir: *Theory of Determinants*, Vol. II, Page 354). Rouche's expressions for Legendre's polynomials are proved by us in another paper.

$$\begin{vmatrix} 1 & \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{2n+1} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \cdots & \cdots & \frac{1}{4n+1} \end{vmatrix} = \left(1\cdot\frac{1}{5}\cdot\frac{1}{9}\cdot\frac{1}{13}\cdots\frac{1}{4^{n+1}}\right)\times \left\{\left(\frac{1\cdot 2}{1\cdot 3}\right)^{n}\left(\frac{3\cdot 4}{5\cdot 7}\right)^{n-1}\cdots\times \left(\frac{2n-1\cdot 2n}{4n-3\cdot 4n-1}\right)^{1}\right\}^2.$$

$$\begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n+1} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots & \frac{1}{2n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \cdots & \cdots & \frac{1}{4n-1} \end{vmatrix} = \left(\frac{1}{3}\cdot\frac{1}{7}\cdot\frac{1}{11}\cdots\frac{1}{4n-1}\right)\times \left\{\left(\frac{2\cdot 3}{3\cdot 5}\right)^{n-1}\left(\frac{4\cdot 5}{7\cdot 9}\right)^{n-2}\cdots \left(\frac{2n-2\cdot 2n-1}{4n-5\cdot 4n-3}\right)^{1}\right\}^2$$

§. 7. Hitherto the quantities A_n have been numbers, but they also occur in the theory as functions of a variable. As examples, we give below without proof the following interesting results relating to Legendre's, Euler's and Bernoulli's polynomials.

(a) Write $a_1 = 1, a_2 = x, a_3 = 0, a_4 = 1, a_5 = x, a_6 = 0; \ldots\ldots$ we find that ${}_1A_n = (1+x)^{n-1}$.

(b) Write $a_1 = \frac{x+1}{2}, a_2 = \frac{x-1}{2}$,

$$a_3 = \frac{x+1}{2} = a_{2n-1},\ a_4 = \frac{x-1}{2} = a_{2n}.$$

$$\text{Then} \quad {}_1A_n = \frac{P_{n+1}(x) - P_{n-1}(x)}{(2n+1)(x-1)} = \frac{1}{x-1}\int_1^x P_n(x)\,dx,$$

where $P_n(x)$ is Legendre's ploy nomial of the nth degree.

From (15), (16) and (17) above, we obtain the following interesting results relating to determinants.

$$\text{Let} \quad A_n \text{ stand for } \int_1^x P_n(x)\,dx, = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$$

$$\text{Then} \qquad A_0 = \frac{P_1(x) - P_{-1}(x)}{1} = (x-1).$$

$$\begin{vmatrix} A_0 & A_1 & A_2 & \cdots & A_n \\ A_1 & A_2 & \cdots & \cdots & A_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & A_{n+1} & & \cdots & A_{2n} \end{vmatrix} = \frac{(x+1)^{\frac{n(n+1)}{2}} \cdot (x-1)^{\frac{(n+1)(n+2)}{2}}}{2^{n(n+1)}}.$$

$$\begin{vmatrix} A_1 & A_2 & \cdots & A_n \\ {}_2\ A_3 & & \cdots & A_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ A_n & A_{n+1} & \cdots & A_{2n-1} \end{vmatrix} = \frac{(x^2-1)^{\frac{n(n+1)}{2}}}{2^{n^2}}.$$

$$\begin{vmatrix} A_2 & A_3 & \cdots & A_{n+1} \\ A_3 & A_4 & \cdots & A_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n+1} & A_{n+2} & \cdots & A_{2n} \end{vmatrix} = \frac{(x^2-1)^{\frac{n(n+1)}{2}}}{2^{n^2+n+1}} \cdot \left[(x+1)^{n+1} - (x-1)^{n+1}\right]$$

(*c*) Form a table with $a_1 = x(1-x)$, $a_2 = 1^2$, $a_3 = (1+x)(2-x)$,

$$a_4 = 2^2; \quad a_{2n-1} = (n-1+x)(n-x); \quad a_{2n} = n^2.$$

Then ${}_1A_1 = -2\psi_2(x)$, ${}_1A_2 = 2\psi_4(x)$, ... ${}_1A_n = (-1)^n\, 2\psi_{2n}(x)$,...

where $\psi_n(x)$ is the co-efficient of $\dfrac{t^n}{n!}$ in the expansion of $\dfrac{e^{xt}}{e^t+1}$.

$$2^{n+1}\begin{vmatrix} \frac{1}{2} & \psi_2 & \psi_4 & \cdots & \psi_{2n} \\ \psi_2 & \psi_4 & \psi_6 & \cdots & \psi_{2n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_{2n} & \psi_{2n+2} & \cdots & \cdots & \psi_{4n} \end{vmatrix} = (1^n\, 2^{n-1}\, 3^{n-2} \ldots n^1)^2\, x^n \times (1-x)(2-x)\ldots(n-x) \times (1^2-x^2)^{n-1} (2^2-x^2)^{n-2} \ldots (\overline{n-1}^2-x^2)^1.$$

$$(-)^n 2^n \begin{vmatrix} \psi_2 & \psi_4 & \dots & \psi_{2n} \\ \psi_4 & \psi_6 & \dots & \psi_{2n+2} \\ \dots & \dots & \dots & \dots \\ \psi_{2n} & \psi_{2n+2} & \dots & \psi_{4n-2} \end{vmatrix} = [1^{n-1} 2^{n-2} 3^{n-3} \dots (n-1)^1]^2 \times x^n (1-x)(2-x) \dots (n-x) (1^2-x^2)^{n-1} (2^2-x^2)^{n-2} \dots (\overline{n-1}^2 - x^2)^1.$$

(*d*) Write $a_1 = 1^2 \dfrac{2x(2-2x)}{1 \cdot 3}$, $a_2 = \dfrac{2^2(1+2x)(3-2x)}{3 \cdot 5}, \dots$

$$a_3 = \frac{3^2(2+2x)(4-2x)}{5 \cdot 7} \dots a_n = \frac{n^2(\overline{n-1}+2x)(n+1-2x)}{(2n-1)(2n+1)}, \dots$$

Then ${}_1A_1 = \dfrac{-2^3 \phi_3(x)}{3(2x-1)}$, ${}_1A_2 = \dfrac{2^5 \phi_5(x)}{5(2x-1)}, \dots$

$${}_1A_n = \frac{(-1)^n 2^{2n+1} \phi_{2n+1}(x)}{(2n+1)(2x-1)}$$

where $\phi_n(x)$ is Bernoulli's polynomial of the nth degree, and is the coefficient of $\dfrac{t^n}{n!}$ in the expansion of $t \dfrac{e^{xt}-1}{e^t-1}$.

$$2^{n(n+1)} \begin{vmatrix} x-\frac{1}{2} & \frac{\phi_3}{3} & \frac{\phi_5}{5} & \dots & \frac{\phi_{2n+1}}{2n+1} \\ \frac{\phi_3}{3} & \frac{\phi_5}{5} & \frac{\phi_7}{7} & \dots & \frac{\phi_{2n+3}}{2n+3} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\phi_{2n+1}}{2n+1} & \frac{\phi_{2n+3}}{2n+3} & \dots & \dots & \frac{\phi_{4n+1}}{4n+1} \end{vmatrix}$$

$$= \left(x-\frac{1}{2}\right)^{n+1} . [(1\cdot 2)^n \times (3 \cdot 4)^{n-1} \dots (\overline{2n-1}.2n)^1]^2 \times \frac{1}{1 \cdot 5 \cdot 9 \dots (4n+1)} \times \frac{1}{[(1\cdot 3)^n (5 \cdot 7)^{n-1} \dots (\overline{4n-3} \cdot \overline{4n-1})^1]^2}$$

$$\times \{x(1-x)\}^n \{(1+x)(2-x)\}^{n-1} \dots \{(\overline{n-1}+x)(n-x)\}^1$$

$$\times \{(1+2x)(3-2x)\}^n \{(3+2x)(5-2x)\}^{n-1} \dots \{(\overline{2n-1}+2x)(2n+1-2x)\}^1.$$

$$(-)^n 2^{n(n-1)} \begin{vmatrix} \frac{\phi_3}{3} & \frac{\phi_5}{5} & \cdots & \frac{\phi_{2n+1}}{2n+1} \\ \frac{\phi_5}{5} & \frac{\phi_7}{7} & \cdots & \frac{\phi_{2n+3}}{2n+3} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\phi_{2n+1}}{2n+1} & \cdots & \cdots & \frac{\phi_{4n-1}}{4n-1} \end{vmatrix}.$$

$$= (x-\tfrac{1}{2})^n \frac{[1^n \cdot (2\cdot 3)^{n-1} \ldots (2n-2\cdot 2n-1)]^2}{3\cdot 7\cdot 11 \ldots (4n-1)} \times \frac{1}{[1^n (3\cdot 5)^{n-1} \ldots (4n-5\cdot 4n-3)^1]^2}$$

$$\times \{x(1-x)\}^n \{(1+x)(2-x)\}^{n-1} \ldots \{(n-1+x)(n-x)\}^1 \times \{(1+2x)(3-2x)\}^{n-1} \{(3+2x)(5-2x)\}^{n-2} \ldots \{(2n-3+2x)\cdot(2n-1-2x)\}^1.$$

[*To be concluded.*]

APPENDIX I. Table for $a_n = 1$.

1	2	5	14	42	132	439	1450	4912	16936
1	3	9	28	90	297	1011	3462	12024	42270
1	4	14	48	165	572	2012	7112	25334	90920
1	5	20	75	275	1001	3650	13310	48650	178400
1	6	27	110	429	1638	6198	23316	87480	327976
1	7	35	154	637	2548	10006	38830	149576	
1	8	44	208	910	3808	15514	62096		
1	9	54	273	1260	5508	23266			
1	10	65	350	1700	7752				
1	11	77	440	2244					
1	12	90	544						
1	13	104							
1	14								
1									

N. B.—The process of multiplying by the numbers of the first column, *viz.*, by unity is omitted.

2, 5, 14, ... are the coefficients in the expansion of $\frac{2}{x} J_2(2x)$.

3, 9, 28, ... " " $\frac{3}{x} J_3(2x)$.

4, 14, 48, ... " " $\frac{4}{x} J_4(2x)$.

1, 2, 5, 14, " " $\frac{1}{x} J_1(2x)$.

SHORT NOTES

On the Product of all Numbers less than N and Prime to it.

Let $\pi\, d\,(N)$ denote the product of all the positive integers less than N and prime to it.

I. If N is prime, $\pi\, d\,(N)$ is evidently $(N-1)\,!$

II. If N is composite, we shall first find the product of all the integers less than N and not prime to N.

Let $p, q, r, s, t, \ldots\ldots$ be the different primes which divide N ; *i.e.* let

$$N = p^{\alpha} \; q^{\beta} \; r^{\gamma} \; s^{\delta} \;\ldots\ldots$$

Consider the series of integers $1,2,3\ldots\ldots\ldots\ldots N-1$. Of these the following are multiples of p: $1.p,\ 2.p, \ldots\ldots \left(\frac{N}{p}-1\right).p.$

$\therefore$ The product of these $= \left(\frac{N}{p}-1\right)!\; p^{\frac{N}{p}-1}$.

Similarly the product of all the multiples of $q = \left(\frac{N}{q}-1\right)!\; q^{\frac{N}{p}-1}$.

Hence the product of all the multiples of p, all the multiples of q, &c., in the series is given by

$$P_1 = \pi \left\{ \left(\frac{N}{p}-1\right)!\, p^{\frac{N}{p}-1} \right\}.$$

In the same series there are $\left(\frac{N}{pq}-1\right)$ multiples of pq and their product is

$$\left\{ \left(\frac{N}{pq}-1\right)!\; \left(pq\right)^{\frac{N}{pq}-1} \right\}.$$

Similarly for multiples of $pr, ps, \ldots qr, \ldots$ (taking all the binary products of $p, q, r, \ldots\ldots$)

Hence the product of all such multiples is given by

$$P_2 = \pi \left\{ \left(\frac{N}{pq}-1\right)!\; \left(pq\right)^{\frac{N}{pq}-1} \right\}.$$

Similarly the product of all the multiples of the ternary products $pqr, pqs, prs, qrs, \ldots\ldots$ is

$$P_3 = \pi \left\{ \left(\frac{N}{pqr}-1\right)!\; \left(pqr\right)^{\frac{N}{pqr}-1} \right\}.$$

And so on.

Now consider the product $P = P_1 \div P_2 \times P_3 \div P_4 \ldots\ldots$

Take any number x which is less than N and not prime to it. It will contain as factors a certain number (k, say) of the different primes $p, q, r, s, \ldots$ Now x will occur k times in the enumeration of the multiples of $p, q, r, \ldots$ taken one at a time. Hence the index of the power of x in P_1 is k. Again x will occur ${}_kC_2$ times in the enumeration of the multiples of the binary products $pq, pr, \ldots$; that is, the index of the power of x in P_2 is ${}_kC_2$; and so on. Hence the index of the power of x in P is ${}_kC_1 - {}_kC_2 + {}_kC_3 - \ldots\ldots = 1$; so that every integer which has a factor in common with N is contained, without repetition or omission, in P.

Hence the product of all the positive integers less than N and not prime to it is P

$$\frac{\pi\left\{\frac{N}{p}-1!\,p^{p-1}\right\}\pi\left\{\frac{N}{pqr}-1!\,(pqr)^{\frac{N}{pqr}-1}\right\}\pi\left\{\frac{N}{pqrst}-1!\,(pqrst)^{\frac{N}{pqrst}-1}\right\}}{\pi\left\{\frac{N}{pq}-1!\,(pq)^{\frac{N}{pq}-1}\right\}\pi\left\{\frac{N}{pqrs}-1!\,(pqrs)^{\frac{N}{pqrs}-1}\right\}\ldots}$$

Now the index of the power of p in P is evidently

$$\left(\frac{N}{p}-1\right)-\left\{\left(\frac{N}{pq}-1\right)+\left(\frac{N}{pr}-1\right)+\left(\frac{N}{ps}-1\right)+\ldots\right\}$$

$$+\left\{\left(\frac{N}{pqr}-1\right)+\left(\frac{N}{pqs}-1\right)+\left(\frac{N}{prs}-1\right)+\ldots\right\}-\ldots\ldots$$

$$=\frac{N}{p}\left\{1-\left(\frac{1}{q}+\frac{1}{r}+\frac{1}{s}\ldots\right)+\left(\frac{1}{qr}+\frac{1}{qs}+\frac{1}{rs}\ldots\right)\ldots\right\}-$$

$$\left\{1-{}_{N-1}C_1+{}_{N-1}C_2-\ldots\ldots\right\}$$

$$=\frac{N}{p}\left(1-\frac{1}{q}\right)+\left(1-\frac{1}{r}\right)\ldots+0$$

$$=\frac{N\,(q-1)\,(r-1)\,\ldots\ldots}{pqr\ldots\ldots}=\lambda \text{ say.}$$

Hence

$$P=\pi\left\{\frac{\left(\frac{N}{p}-1\right)!\;\left(\frac{N\,t}{pqr}-1\right)!\ldots\ldots p^{\lambda}}{\left(\frac{N}{pq}-1\right)!\;\left(\frac{N}{pqrs}-1\right)!\;\ldots\ldots}\right\}$$

Now the product of all the integers less than N is $(N-1)!$; hence,

finally, the product of all the integers less than N and prime to it

$$=(N-1)!! \div P=(N-1)!\,\pi\left[\left\{\left(\frac{N}{pq}-1!\right)\left(\frac{N}{pqrs}-1\right)!\ \ldots\ldots\right\}\div\right.$$

$$\left.\left\{\left(\frac{N}{p}-1!\right)\left(\frac{N}{pqr}-1\right)!\ \ldots p^{\lambda}\ldots\right]\right. \quad \text{(A)}$$

Ex. (1) If $N = 12$, then $p = 2, q = 3, \alpha = 2, \beta = 1$ and

$$\pi\, d\,(12) = \frac{11!\ 1!}{53!\ 2^4 3^2}.$$

(2) If N = 72, then $p = 2, q = 3, \alpha = 3, \beta = 2$ and

$$\pi\, d\,(72) = \frac{71!\ 11!}{35!\ 23!\ 2^{24}3^{12}}$$

The theorem is some times stated *incorrectly* in the following form:—

If $N = abcd\ldots\ldots k$, where $a, b, c, \ldots\ldots$, are prime to each other, then

$$\pi\, d\,(\mathrm{N}) = (N-1)!\,\pi \left|\frac{\frac{N}{ab}-1!\ \frac{N}{abcd}-1!\ldots\ldots}{\frac{N}{a}-1!\frac{N}{abcd}-1!\ \ldots\ldots a^{(b-c)(c-1)\ldots(k-1)}}\right|$$

Thus in the Conv. and Caius Coll. Exam. 1882, (quoted in Chrystal's *Algebra*, Vol. II, p. 547) the following question was set :—

If $N = abc$, where a, b, c are prime to each other, then the product of all the numbers less than N and prime to N is

$$(abc-1)!\,\pi\ \{(a-1)!\,(bc-1)!\,a^{(b-1)(c-1)}\}.$$

That this theorem is wrong can be verified by taking any particular case, *e.g.*, let $a = 3, b = 4, c = 5$, then N = 60. Then according to this theorem $\pi\, d\,(\mathrm{N}) = \dfrac{59!\ 2!\ 3!\ 4!}{19!\ 14!\ 11!\ 3^{12}4^8 5^6}$

Now the indices of the powers of 2 contained in 59 !, 2 !, 3 !, 4 !, 19 !, 14 and 11 ! are respectively 54, 1, 1, 3, 16, 11 & 8, so that the index of the power of 2 in $\pi\, d$ (60) is $(54+1+1+1)-(16+11+8+16)=8$. Since 2 is not prime to 60, no number prime to 60 can contain any power of 2 as a factor. Hence $\pi\, d$ (60) cannot contain any power of 2. Hence the theorem is obviously wrong.

The correct answer in the case considered is given by (A). Thus, here $p = 2$, $q = 3$, $r = 5$, $\alpha = 2$, $\beta = 1$, $\gamma = 1$, so that by (A),

$$\pi\, d\,(60) = \frac{59\,!\,9\,!\,5\,!\,3\,!}{29\,!\;19\,!\;11\,!\;2^{16}\,3^{8}\,5^{4}}.$$

Now the numerator of this fraction is $= (2^{54}\,3^{27}\,5^{13}\,7^{9}\,11^{5}\,13^{4}\,17^{3}\,19^{3}\,23^{2}\,29^{2}.\,31.\,37.\,41.\,43.\,47.\,53.\,59)\,(2^{7}\,3^{4}\,5.\,7)\,(2^{3}\,3.\,5)\,(2.3)$ and the denominator $= (2^{25}3^{13}5^{6}7^{4}\,11^{2}\,13^{2}\,17.\,19.\,23.\,29)\,(2^{16}3^{8}5^{3}7^{2}11.\,13.\,17.\,19)\,(2^{8}\,3^{4}\,5^{2}\,7.\,11)\,2^{16}3^{8}5^{4}$ so that $\pi\, d\,(60) = 7.^{3}11.\,13.\,17.\,19.\,23.\,29.\,31.\,37.\,41.\,43.\,47.\,53.\,59.$

That this is the correct answer can be easily verified; for the integers less than 60, and prime to it are 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 52, 59.

Question 1013 of the J. I.M.S. (Vol. X, Dec. 1918, p. 492) is similarly wrongly stated.

N. B. MITRA.

Three Fundamental Formulæ.

[The formulæ here proved have been assumed in our Paper on Determinants. (Vide : J.I.M.S., Vol. XIV, p. 55, §§ 1 and 3.)]

1. Let $y = \sec x = \sum E_n \frac{x}{2n!}$, where E_n is the nth Eulerian number, so that

$$\left.\frac{d_{2n}y}{dx^2}\right]_{x=0} = E_n.$$

Now let us obtain the expansion of the nth differential coefficient of $\sec x$ as the product of $s = \sec x$ and a power series in $t = \tan\ x$.

It is easily seen that

$$\frac{d^4y}{dx^4} = s\ \{\ 4!\, t^4 + 2!\ (1^2 + 2^2 + 3^2)\, t^2 + (1^2 + 2^2)\}$$

$$\therefore \qquad E_2 = 1^2 + 2^2.$$

$$\frac{d^6y}{dx^6} = s\ \{6!\, t^6 + 4!\, t^4 \times (1^2 + 2^2 + 3^2 + 4^2 + 5^2)$$

$$+ 2!\, t^2\ \{1^2\,(1^2 + 2^2) + 2^2\,(1^2 + 2^2 + 3^2)$$

$$+ 3^2\,(1^2 + 2^2 + 3^2 + 4^2)\}$$

$$+\ \{1^2\,(1^2 + 2^2) + 2^2\,(1^2 + 2^2 + 3^2)\}$$

$$\therefore E_3 = 1^2\,(1^2 + 2^2) + 2^2\,(1^2 + 2^2 + 3^2) = \Sigma 1^2\ \Sigma 2^2\ \Sigma 3^2.$$

Similarly it is easily seen that

$$\frac{d^8y}{dx^8} = s\Big\{\, 8!\, t^8 + 6!\,\Sigma 7^2\, t^6 + 4!\, t^4\, \Sigma 5^2\, \Sigma 6^2$$

$$+ 2!\, t^2\, \Sigma 3^2\, \Sigma 4^2\, \Sigma 5^2 + \Sigma 1^2\, \Sigma 2^2\, \Sigma 3^2\, \Sigma 4^2\,\}\ \ldots$$

$$\therefore E_4 = \Sigma 1^2\ \Sigma 2^2\ \Sigma 3^2\ \Sigma 4^2.$$

Now assume that

$$\frac{d^ny}{dx^n} = s\ \Big\{ n!\, t^n + \overline{n-2}!\ t^{n-2}\ \Sigma(n-1)^2$$

$$+ \overline{n-4}.!\ t^{n-4}\ \Sigma(n-3)^2\ \Sigma(n-2)^2 + \ldots$$

$$+ \overline{n-2r}!\ t^{n-2r}\ \Sigma(n-2r+1)^2 \ldots\ldots$$

$$\Sigma(n-r+1)^2\ \Sigma(n-r)^2 + \ldots \} \qquad (1\cdot 1)$$

the last term being $\Sigma 2^2\ \Sigma 3^2\ \ldots\ \Sigma \frac{(n+1)^2}{2}\ t$, if n is odd,

and $$\Sigma 1^2\ \Sigma 2^2 \ldots \Sigma \left(\frac{n}{2}\right)^2 = E_{\frac{n}{2}}, \text{ if } n \text{ is even.} \qquad (1)$$

Differentiating, writing $\frac{dt}{dx} = 1 + t^2$, and collecting the coefficients of t^{n-2r+1}, we find this coefficient is

$$\begin{aligned} & \underline{n-2r}!\ \Sigma(n-2r+1)^2\ \Sigma(n-2r+2)^2 \ldots \Sigma(n-r)^2 \\ & + (n-2r+2)\ \underline{n-2r+2}!\ \Sigma(n-2r+3)^2 \\ & \quad \Sigma(n-2r+4)^2 \ldots \Sigma(n-r+1)^2 + \underline{n-2r}!\ (n-2r) \\ & \quad \Sigma(n-2r+1)^2\ \Sigma(n-2r+2)^2 \ldots \Sigma(n-r)^2. \\ & = \underline{n-2r+1}!\ \{ (n-2r+{}^2\ \Sigma 2)(n-2r+3)^2 \ldots \\ & \quad \Sigma(n-r+1)^2 + \Sigma(n-2r+1)^2 \ldots \Sigma(n-r)^2 \}. \\ & = \underline{n-2r+1}!\ \Sigma(n-2r+2)^2\ \Sigma(n-2r+3)^2 \ldots \Sigma(n-r+1)^2 \qquad (1{\cdot}4) \end{aligned}$$

Hence by induction, the formulæ (1.) and (1.1) follow immediately.

[NOTE :—The formula (1.1) is elegantly expressed by the table in Table I for Euler's numbers, which was kindly suggested to us by Mr. K. B. Madhava.

The first column contains the squares of the natural numbers, *viz.* 1, 4, 9, 16,...The second is obtained from the first by an obvious method of addition, *e.g.* $5 = 1^2 + 2^2$, $14 = 1^2 + 2^2 + 3^2$,...The third is obtained by multiplying the numbers in the second column by the corresponding numbers in the first. *e.g.* 56 = 4.14, 270 = 9·30,...The fourth is obtained from the third by addition as before and the process is repeated. We obtain Euler's numbers in pairs at the top.]

2. In a similar manner, since

$$\tan x = \Sigma b_n \cdot \frac{x^{2n-1}}{\underline{2n-1}!}$$

where $b_n = 2^{2n}(2^{2n}-1)\frac{B_n}{2n} = n^{\text{th}}$ prepared Bernoullian number (an integer), we can prove that

$$\begin{aligned} \frac{dt}{dx} &= n!\ t^{n+1} + \underline{n-2}!\ t^{n-1}\ \Sigma(n-1.n) \\ & + \underline{n-4}!\ t^{n-3}\ \Sigma(n-3.n-2)\ \Sigma(n-2.n-1) + \ldots \end{aligned}$$

$$+ (n-2r)!\, t^{n-2r+1}\, \Sigma(n-2r+1,\, n-2r) + \Sigma(n-2r.\, n-2r-1) \ldots \Sigma(n-r.\, n+r+1) + \ldots \quad (2{\cdot}1)$$

the last term being

$$t\ \Sigma(1.2)\, \Sigma(2.3) \ldots \Sigma\left(\frac{n}{2}.\ \frac{n}{2}+1\right), \text{ if } n \text{ is even,}$$

and $$\Sigma(1.2)\, \Sigma(2.3) \ldots \Sigma\left(\frac{n-1}{2}.\ \frac{n+1}{2}\right) = b\,\frac{n+1}{2}, \text{ if } n \text{ is odd.} \quad (2)$$

Hence, if a table is formed exactly as in 2 above with 1.2, 2.3, 3.4, ... in the first column, we obtain the prepared Bernoullians beginning with the second (b_2) in the odd columns of the top row. (See Table II.)

3. Again $$\frac{1}{\cos x - a \sin x} = \sec x \left\{ 1 + \Sigma\, a^n \tan^n x \right\}.$$

Also if we write

$$\frac{1}{\cos x - a \sin x} = 1 + \Sigma\, A_n(a).\ \frac{x^n}{n!}, \quad (3.1)$$

then A_n is obviously a function of a of degree n, odd or even according as n is odd or even, and

$$A_n(a) = \frac{d^n}{dx^n}\left(\frac{1}{\cos x - a \sin x}\right)\Big]_{x=0}.$$

$$= \cos\theta\ \frac{d^n}{dx_n}(\sec\theta), \text{ where } \tan\theta = a,$$

so that

$$A_n(a) = n!\, a^n + (n-2)!\, a^{n-2}\, \Sigma(n-1)^2 + \ldots$$

$$+ (n-2r)!\, a^{n-2r}\, \Sigma(n-2r+1)^2 \ldots \Sigma(n-r)^2 + \ldots\ldots \quad (3{\cdot}2)$$

Hence by rearranging (3.1) in powers of a with the help of (3), we have

$$\sec x \tan^n x = n!\left\{\frac{x^n}{n!} + \frac{x^{n+2}}{n+2!}\, \Sigma(n+1)^2\right.$$

$$+ \frac{x^{n+4}}{n+4!}\, \Sigma(n+1)^2\, \Sigma(n+2)^2 + \ldots\ldots$$

$$\left. + \frac{x^{n+2r}}{n+2r!}\, \Sigma(n+1)^2\, \Sigma(n+2)^2 \ldots \Sigma(n+r)^2 + \ldots\ldots \right\} \quad (3)$$

All the coefficients in (3) are to be found in the even columns of Table 1.

TABLE I.

1	5	5	61	61	1385	1385	50521	50521
4	14	56	331	1324	12284	49136		
9	30	270	1211	10899				
16	55	880						
25								

TABLE II.

2	8	16	136	272	3968	7936	176896	353792
6	20	120	616	3696	28160	168960		
12	40	480	2016	24192				
20	70	1400						
30								

C. KRISHNAMACHARI.

M. BHIMASENA RAO.

SOLUTIONS.

Question 1127.

(K. J. SANJANA, M.A.):—Prove that there are two and only two Tucker circles of a triangle which touch a given straight line. These circles coalesce when the given line is one of the sides of the triangle.

If O and K be the circumcentre and symmedian point of a triangle ABC, T_1 the centre and R_1 the length of the radius of the Tucker circle touching BC, prove that

$$KT_1 : T_1O = b^2 + c^2 - a^2 : b^2 + c^2 + a^2 \text{ and } R_1 : R = bc : (b^2 + c^2).$$

Additional Solution by the Proposer.

An elegant geometrical solution of this question is given by Mr. M. M. Thomas in the February (1922) number of our Journal. The following analytical solution may prove of interest.

As proved in my paper on Tucker Circles printed in the Journal for December 1917, the trilinear equation of a Tucker circle of *anti-parallel intercept* μ is

$$abc\,(\Sigma a\beta\gamma) - \mu\,(\Sigma a\alpha)\,.\,\Sigma\,\{\,(bc - a\mu)\,\alpha\,\} = 0.$$

This may be written in the form

$$\Sigma(\mu^2a^2 - \mu abc)\,\alpha^2 + \Sigma\,\{\,a^2bc - \mu a\,(b^2+c^2) + 2\mu^2bc\,\}\;\beta\gamma = 0.$$

The condition that the straight line $l\alpha + m\beta + n\gamma = 0$ should touch the circle (the tangential equation of the Tucker circle) being written in the form $\Sigma Al^2 + \Sigma 2Fmn = 0$, it will be found that,

$$A = -\tfrac{1}{4}a^2\;\{\,\mu\,(b^2 + c^2) - abc\,\}^2,$$

with similar values for B and C, and that

$$F = \tfrac{1}{4}\,\mu^2bc(-a^4 + a^2b^2 + a^2c^2 + b^2c^2) - \tfrac{1}{4}\mu ab^2c^2\,(b^2+c^2) + \tfrac{1}{4}\,a^2b^3c^3,$$

with similar values for G and H.

Since these values of A, B, C and F, G, H involve μ only to the second power, it follows that when l, m, n are given we get a quadratic equation to determine μ. Hence there cannot be more than two Tucker circles of a triangle touching a given straight line in the plane of the triangle.

When the given line is a side of the triangle, say BC, we have $m = n = 0$, and the condition of tangency reduces to

$$Al^2 = 0, \text{ or }\;\{\,\mu(b^2+c^2) - abc\,\}^2 = 0.$$

Thus the two circles coalesce, the anti-parallel intercept becoming $abc/(b^2 + c^2)$.

In this case, if the ratio $KT_1 : KO$ is denoted by e, we have as shown in the paper cited above

$$\mu = \frac{2(1-e)abc}{a^2+b^2+c^2} = \frac{abc}{b^2+c^2}; \therefore 1-e = \frac{a^2+b^2+c^2}{2b^2+2c^2},$$

so that $$e = (b^2+c^2-a^2)/(2b^2+2c^2).$$

But $$KT_1 : KO = e.$$

$$\therefore KT_1 : T_1O = e : 1-e = b^2+c^2-a^2 : b^2+c^2+a^2.$$

Finally, as proved in the same paper, we have

$$R_1^2 = R^2e^2 + \tfrac{1}{4}\mu^2 = \frac{R^2(b^2+c^2-a^2)^2}{4(b^2+c^2)^2} + \frac{a^2b^2c^2}{4(b^2+c^2)^2}$$

$$= \frac{1}{4(b^2+c^2)^2}\left\{R^2(b^2+c^2-a^2)^2 + R^2 . 16\Delta^2\right\}$$

$$= \frac{R^2 . 4b^2c^2}{4(b^2+c^2)^2};$$

$$\therefore \quad R_1 : R = bc : (b^2+c^2).$$

Question 1145.

(N. Dorai Rajan) :—n rods $OA_1, OA_2, \ldots\ldots OA_n$ are hinged together at O which is a plane joint. Show that the area of the plane polygon $A_1A^2\ldots\ldots A_n$ is a maximum, when the circles on A_1A_2, $A_2A_3, \ldots\ldots A_nA_1$ as diameters have a common orthogonal circle; and that the perimeter is a maximum when all the sides touch a circle. When there are only three rods, can the triangle be constructed with the ruler and compasses?

Geometrical Solution by G. V. Krishnaswami.

Let the Δs OA_3A_4, $OA_4A_5, \ldots\ldots\ldots OA_n A_1$ be kept fixed. Then A_1 and A_3 are fixed points and A_2 can vary its position moving on a circle with O as centre and OA_2 as radius.

Firstly $A_1A_2 + A_2A_3$ will be greatest, if A_1A_2 and A_2A_3 are equally inclined to the tangent to the circle at A_2, or what is the same things to the radius OA_2. Let A_2 take such a position. Now keep the Δs OA_4A_5, $OA_5A_6 \ldots\ldots\ldots OA_1A_2$ fixed. $A_2A_3 + A_3A_4$ can be made greatest by taking OA_3 the bisector of the angle $A_2A^3A_4$. Hence the perimeter is a maximum when OA_1, $OA_2 \ldots\ldots$ are the bisectors of the angles $A_nA_1A^2$, $A_1A_2A_3 \ldots\ldots$; that is, the sides A_1A_2, $A_2A_3 \ldots\ldots$touch a circle with O as centre and the equal altitudes as radius.

Secondly, keeping the Δs OA_3A_4, OA_4A_5, ... OA_nA_1 fixed as before, the quadrilateral $OA_1A_2A_3$ has maximum area if OA_2 is perpendicular to A_1A_3. Hence as before the polygon has maximum area if OA_1, OA_2 ...... are perpendicular respectively to A_nA_2, A_1A_3, A_2A_4 .... O is therefore the radical centre of the circles on A_1A_2, A_2A_3, ... as diameters ; that is, these circles have a common orthogonal circle whose centre is O.

When there are only three rods, O becomes the orthocentre of the triangle $A_1A_2A_3$ if the area is to be a maximum ; and if the perimeter is maximum, O becomes the in-centre of that triangle.

Hence the problem is to construct a triangle given OA, OB, OC (i) when O is the ortho-centre and (ii) when O is the in-centre of the triangle ABC.

Question 1152.

(SELECTED) :—Find the complete primitive of the differential equation

$$9xy^2 \frac{d^2y}{dx^2} - 2 = 0. \quad \text{(Forsyth : } \textit{Diffl. Eqns.}\text{)}$$

Solution by Prof. Wilkinson.

1. Solution of $2a^2 xy^2 \dfrac{d^2y}{dx^2} + 1 = 0.$

Noticing that

$$2a^2 (px-y)\ x \frac{d^2y}{dx^2} + \frac{px-y}{y^2} = 0$$

is exact ; we have, on integration

$$a^2 (px-y)^2 = A + \frac{x}{y} = v^2 \text{ say ;}$$

this gives
$$\frac{dx}{x^2} = \frac{2adv}{(v^2 - A)^2},$$

which can be integrated in the usual way.

2. Similarly, we can integrate $2a^2xy^2 \dfrac{d^2y}{dx^2} - 1 = 0.$

Question 1182.

(B. B. BAGI.) :—The sides taken in order of an n—gon (n being odd) circumscribed to a circle are $a_1, a_2, ... a_n$. Prove that the radius of the inscribed circle is given by

$$\text{(i) } \tan^{-1} \frac{x}{s-a_1-a_3-\ldots-a_{n-2}} + \tan^{-1} \frac{x}{s-a_2-a_n-\ldots-a_{n-1}} + \ldots = \frac{\pi}{2}(n-2)$$

and that the area is determined by

$$\text{(ii)} \tan^{-1}\frac{\Delta}{s(s-a_1-a_2-\ldots-a_{n-2})}+\tan^{-1}\frac{\Delta}{s(s-a_2-a_4-\ldots-a_{n-1})}+\ldots=\frac{\pi}{2}(n-2),$$

where $2s=a_1+a+a+\ldots+a_n$.

Solutions by A. A. Krishnaswami Iyengar, L. N. Subramanian, C. Ranganathan and K. Satyanarayana.

Let $A_1 A_2 \ldots\ldots A_n$ be the polygon and $P_1, P_2, P, \ldots P_n$ the points of contact with the in-circle. Let O be the in-centre.

Now, from the geometry of the figures it is evident that

$$A_r P_r = s - a_{r+1} - a_{r+3} - \ldots\ldots - a_{r-2}$$

and $\tan O\hat{A}_r P_r = \tan\frac{A_r}{2} = \frac{x}{s-a_{r+1}-a_{r+3}-\ldots-a_{r-2}}$

Again, since $x.s = \Delta$,

$$\tan\frac{A_r}{2} = \frac{\Delta}{s(s-a_{r+1}-a_{r+3}-\ldots-a_{r-2})}$$

$$\therefore \quad \frac{A_r}{2} = \tan^{-1}\frac{\Delta}{s(s-a^{r+1}-a_{r+3}-\ldots)}$$

Since $\Sigma A_r = \pi(n-2)$, we get the results

$$\Sigma \tan^{-1}\frac{x}{s-a_{r+1}-a_{r-2}\ldots-a_{r-2}}$$

$$= \Sigma \tan^{-1}\frac{\Delta}{s(s-a_{r+1}-a^{r+3}-\ldots-a_{r-2})} = \frac{\pi}{2}(n-2).$$

Question 1183.

(B. B. Bagi) :—The circles round AQR, BRP, CPQ, where P, Q and R are points in order on the sides BC, CA, AB of a triangle ABC meet in O. If A′, B′, C′ are the middle points of the arcs QOR, ROP, POQ, then show that A′ B′ C′ is a triangle similar to the triangle of the ex-centres of ABC and also that A′ B′ C′ and the in-centre of ABC are concyclic.

Remarks and Solution by A. A. Krishnaswami Aiyangar.

This question can be extended and generalised as follows :—

ABC is a triangle. P, Q, R are points in the sides BC, CA, AB and the circles round AQR, BRP, CPQ meet in O. If A′, B′, C′ are any points on the circles AQR, BRP, CPQ, such that AA′, BB′, CC′ meet in O′, then the five points A′, B′, C′, O, O′ lie on a circle.

$$\text{Now } A'\hat{O}B' = R\hat{O}P - R\hat{O}A' + P\hat{O}B'$$

$$= \pi - B - R\hat{A}A' + P\hat{B}B'$$

$$= \pi - A\hat{B}O' - R\hat{A}A'$$

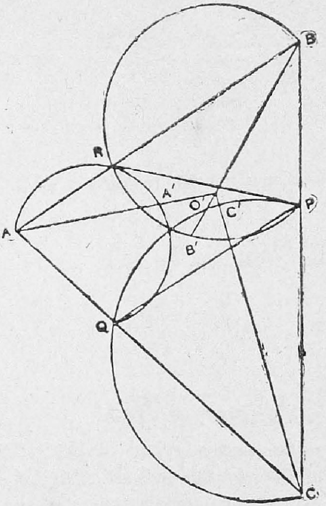
$$= \pi - A'\hat{O}'B'.$$

∴ A′B′ subtends supplementary angles at O and O′

Hence O, A′, B′, O′ are concyclic.

Similarly, it can be shown that A′C′ subtends supplementary angles at O and O′.

Hence the five points A′, B′, C′, C, O′ are concyclic.



Cor. 1. In the triangle A′B′C′, $B'\hat{A}'C'$ is the supplement of the angle BO′C ; $A'\hat{B}'C'$ is the supplement of the angle CO′A ; and $A'\hat{C}'B$ is the supplement of the angle AO′B.

Hence the triangle $A'B'C'$ is similar to the anti-pedal triangle of ABC with respect to O'.

2. In particular, if O' be the in-centre of the triangle ABC, the triangle $A'B'C'$ is aimilar to the ex-central triangle of the triangle ABC, which is Q. 1183.

3. If O' be the circum-centre of the triangle ABC, $A'B'C'$ is similar to the pedal triangle of ABC.

4. If PQR be the pedal triangle of O with respect to ABC, OO' is the diameter of the circum-circle of $A'B'C'$.

5. If O, O' be isogonal conjugates with respect to the triangle ABC, OA', OB', OC', become parallel to QR, RP, PQ, respectively, and thus the perpendicular bisectors OA', OB', OC' become identical with those QR, RP, PQ. Hence the circum-centre of the triangle $A'B'C'$ coincides with that of the triangle PQR.

6. When O,O' are the positive and the negative Brocard points of the triangle ABC, the circle round PQR becomes a Tucker circle of the triangle ABC and its centre, by the previous corollary obviously lies on the perpendicular bisector of OO'. Hence, we get the well-known theorem that the perpendicular bisector of the straight line joining the Brocard points is the locus of the centre of Tucker's system of circles.—(*Vide* : §§ 36, 37, p. 73, McClelland's *Geometry of the Circle*.)

When the circle PQR becomes a Triplicate-ratio circle, $A'B'C'$ becomes the first Brocard triangle of ABC and we get the theorem that the centre of the triplicate ratio circle is also the centre of the Brocard circle of the triangle ABC.

In conclusion, we may remark that if P',Q',R' be another set of points on the sides of the triangle ABC such that the circles round $AQ'R'$, $BR'P'$, $CP'Q'$, meet in O' and A'',B'',C'' points on these circles such that AA'',BB'',CC'' meet in O, then the following results are easily inferred :—

(i) the eight points A',B',C', A'',B'',C'', O,O' lie on the same circle.

(ii) the pairs of straight lines OP, $O'P'$; OQ, $O'Q'$, ; OR, $O'R'$ are equally inclined to BC, CA, AB respectively.

(iii) the perpendicular bisectors of PP', QQ', RR', are concurrent at the centre of the circle $A'B'C'$.

(iv) the points of intersection of the pairs (OP, $O'P'$), (OQ, $O'Q'$,)and (OR,$O'R'$) lie on a circle passing through the centre of the circle $A'B'C'$.

(v) P,Q,R, P',Q',R', will lie on a Tucker-circle of the triangle ABC, provided O,O' are respectively the positive and the negative Brocard points of the triangle ABC.

Question 1184.

(G. S. MAHAJANI) :—In any triangle, we know that

$$c^2 = a^2 + b^2 - 2\,ab\cos C.$$

Similarly, in any polygon of n sides $(a_1\, a_2 \ldots\ldots a_n)$

$$a_n^2 = \sum_1^{n-1} a_r^2 - 2\sum a_r\, a_s \cos \widehat{a_r\, a_s}$$

r and s being unequal and taking all integral values from 1 to $n-1$.

Symmetrically, the sides and angles of a polygon are connected by the following relation :—

$$\sum_1^{n} a_r^2 - 2\sum a_r\, a_s \cos \widehat{a_r\, a_s} = 0,$$

r, s being unequal and taking all integral values from 1 to n.

Solution and Remarks by A. A. Krishnaswami Aiyangar and several others.

See Arts. 127 and 128, Hobson's *Plane Trigonometry*, 4th Edn., where the result given above is proved, with only a change in sign due to the adoption of a different notation. The formula, however, can be easily proved by induction thus : Denoting the vertices of the $n+1$ sided polygon by $A_1\, A_2 \ldots\ldots A_{n+1}$ and the sides $A_1 A_2$, $A_2 A_3$, …… etc., in order by $a_1, a_2, \ldots\ldots a_{n+1}$, we may write

$$a^2_{n+1} = a_1^2 + x^2 + 1 - 2\,a_1\, y,$$

where $x = A_2\, A_{n+1}$ and y is the projection of x on a_1,

But this projection $=$ the sum of the projections of $a_2, a_3 \ldots a_{n+1}$ on a_1

$$= \sum_2^{n} a_r \cos \widehat{a_1\, a_r}$$

$$\text{and } x^2 = \sum_2^{n} a_r^2 - 2\sum a_r\, a_s \cos \widehat{a_r\, a_s},$$

r and s being unequal and taking all integral values from 2 to n.

$$\therefore \quad a^2_{n+1} = \sum_1^{n} a_r^2 - 2 \sum a_r\, a_s \cos \widehat{a_r\, a_s},$$

which proves the formula.

Question 1189.

(F. B. FREEMAN) :—From a point P tangents are drawn to a given ellipse and to all confocal conics. Shew that the locus of the points of contact of these tangents is the same as the locus of the foot of the perpendicular from P upon the chord of contact. Explain the significance of this property.

Solution by A. Mahalingam.

Let the co-ordinates of P be (x', y') and let the confocals be represented by

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

The points of contact of the tangents from P lie on the chord of contact

$$\frac{x\,x'}{a^2+\lambda} + \frac{y\,y'}{b^2+\lambda} = 1. \qquad \text{(A)}$$

Since they also lie on the confocal, $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$, (B)

the eliminant of λ between the equations (A) and (B) gives the locus of the points of contact.

$$\text{(A)}-\text{(B)} = \frac{x\,(x-x')}{a^2+\lambda} + \frac{y\,(y-y')}{b^2+\lambda} = 0,$$

whence $$\lambda = \frac{-a^2\,y\,(y-y') - b^2\,x\,(x-x')}{x\,(x-x') + y\,(y-y')}$$

Substituting this value of λ in equation (A) after a few easy transformations, it reduces to $\dfrac{x}{y-y'} + \dfrac{y}{x-x'} = \dfrac{a^2-b^2}{x'\,y - x\,y'}$,

which is the locus of the points of contact.

Let us also find the locus of the foot of the perpendicular from P on the chord of contact.

The equation of a line perpendicular to A and passing through P is

$$\frac{x\,y'}{b^2+\lambda} - \frac{y\,x'}{a^2+\lambda} = \frac{x'\,y'\,(a^2-b^2)}{(a^2+\lambda)\,(b^2+\lambda)} \qquad \text{(C)}$$

The eliminant of λ between (C) and (A) gives the locus required.

From equation (C), $\lambda = \dfrac{x'\,y'\,(a^2-b^2) - a^2\,x\,y' + b^2\,x'\,y}{x\,y' - x'\,y}$

On substituting this value of λ in equation A, it reduces to

$$\frac{x}{y-y'} + \frac{y}{x-x'} = \frac{a^2-b^2}{x'y - xy'}$$

which is identical with the locus of the point of contact.

Question 1190.

(N.P. PANDYA) :—It is required to illustrate by putting together a minimum number of card-board pieces, that the complements of a parallelogram are equal. The point through which the parallels are drawn is at a distance of 1·5 in. from the vertex on the longer diagonal of a parallelogram whose angle is 60° and whose sides are 4 in. and 6 in. respectively. Give a description of all pieces required for this purpose.

Solution by G.V. Vasudevasastry.

Let ABCD be the given parallelogram having AB = 4″, AD = 6″ and the angle BAD = 60°. Along the longer diagonal AC, mark off AO = 1·5″. Draw EOF and GOH parallel to AD and AB respectively through O, the former meeting AB and DC in E and F respectively and the latter meeting AD and BC in G and H respectively.

Now it is required to show by illustrating with minimum number of card-board pieces that the complements EOHB and OGDF are equal.

$$AC^2 = 6^2 + 4^2 + 2.6.4.\tfrac{1}{2} = 76.$$

$$\therefore \quad AC = \sqrt{76}.$$

$$\text{Now} \quad \frac{AE}{4} = \frac{EO}{6} = \frac{AO}{AC} = \frac{1\cdot5}{\sqrt{76}} = \delta, \text{ say.}$$

Then we have

$$AE = 6\delta, \; EO = 9\delta.$$

$$\therefore \quad EB = 4 - 6\delta \; \& \; GD = 6 - 9\delta.$$

Now cut the two pieces EOHB and GOFD from the parallelograms. Place the piece EOHB on the piece OGDF, having the edge OE along OG. The edge OH falls along OF and let K be the pt. on OF corresponding to the pt. H. Now cut the card-board OHBE along the line GD in the new position. What is left will be a parallelogram with sides 4—6δ and 3δ. The portion remaining in the parallelogram OGDF is KFDL, where KL is drawn through K parallel to AB, meeting AD in L. The length of the sides in this case are, 2—3δ and 6δ. By cutting into two halves the remaining piece in the former case (4—6δ by 3δ), we can superpose on the parallelogram KFDL and thus show that the complements are equal. The number of card-board pieces required are three.

(1) 4—6δ by 6δ, (2) 2—3δ by 3δ, (3) 2—3δ by 3δ.

[The property is simply illustrated by means of the triangular pieces ABC, AEO, OHC and their congruents CDA, OGA, CFO.

For OHBE = ABC — AEO — OHC,
and OGDF = CDA — OGA — CFO. Ed.]

Question 1194.

(PROF. SANJANA) :—If $f(x)$ denote

$$x^n + 2p_1 x^{n-1} + 2^2 p_2 x^{n-2} + \ldots\ldots + 2^{n-1} p^{n-1} x + 2^n p^n$$

prove that the result of eliminating x from the two equations

$$f(y) + \frac{x^2}{2!} f_2(y) + \frac{x^4}{4!} f_4(y) + \ldots\ldots = 0,$$

$$f_1(y) + \frac{x^2}{3!} f_3(y) + \frac{x^4}{5!} f_5(y) + \ldots\ldots = 0,$$

is the same as that of eliminating $y_2, z_1, z_2 \ldots\ldots z_{n-2}$ from the following n equations—

$$y + z_1 + p_1 = 0,$$
$$yz_1 + y_2 + z_2 - p_2 = 0,$$
$$yz_2 + y_2 z_1 + z_3 + p_3 = 0, \ldots$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$yz_{n-2} + y_2 z_{n-3} - (-1)^{n-1} p_{n-1} = 0,$$
$$y_2 z_{n-2} - (-1)^n p_n = 0.$$

[$f_1, f_2, f_3 \ldots\ldots$ denote derived functions of f with respect to y.]

Solution by K. Satyanarayana.

By Taylor's theorem

$$f(y+x) = f(y) + xf_1(y) + \frac{x^2}{2!} f_2(y) + \ldots + \frac{x^n}{n!} f_n(y).$$

and $f(y-x) = f(y) - xf_1(y) + \frac{x^2}{2!} f_2(y) + \ldots + (-1)^n \frac{x^n}{n!} f_n(y).$

We may take the following as the equations to eliminate x from

$$\left.\begin{aligned}(y+x)^n + 2p_1 (y+x)^{n-1} + 2^2 p_2 (y+x)^{n-2} + \ldots + 2^n p_n = 0,\\ (y-x)^n + 2p_1 (y-x)^{n+1} + 2^2 p_2 (y-x)^{n-2} + \ldots + 2^n p_n = 0.\end{aligned}\right\} \quad (A)$$

The second set of given equations may be written

$$2^n p_n = (-1)^n \cdot 2^n \quad [\quad y_2 z_{n-2}]$$

$$2^{n-1} p_{n-1} (y \pm x) = (-1)^{n-1} 2^{n-1} [yz_{n-2} + y_2 z_{n-3}] (y \pm x)$$

$$2^{n-2} p_{n-2} (y \pm x)^2 = (-1)^{n-2} 2^{n-2} [yz_{n-3} + y_2 z_{n-4} + z_{n-2}] (y \pm x)^2$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$2^2\, p_2\, (y\pm x)^{n-2} = (-1)^2\; 2^2\; [yz_1 + y_2 + z_2]\,(y\pm x)^{n-2}$$

$$2\, p_1\, (y\pm x)^{n-1} = (-1)^1\; 2\; [y + z_1]\,(y\pm x)^{n-1}$$

$$(y\pm x)^n = (y\pm x)^n\,.$$

Sum of L. H. S. gives the equations (A) according as + or — sign is taken, provided it be shown that sum of R. H. S. is zero for either sign.

R. H. S. =

$$\{[(y\pm x)^n - 2y\,(y\pm x)^{n-1}] + (-1)^1 . 2z_1\,[(y\pm x)^{n-1} - 2y\,(y\pm x)^{n-2}]$$

$$+ (-1)^2 . 2^2 z_2\,[(y\pm x)^{n-2} - 2y\,(y\pm x)^{n-3}] + \ldots\ldots$$

$$+ (-1)^{n-2}\, 2^{n-2}\, z_{n-2}\,[(y\pm x)^2 - 2y\,(y\pm x)]\,\}$$

$$+ (-1)^2\, 2^2 y_2\; \{(y\pm x)^{n-2} - 2z_1\,(y\pm x)^{n-3} + (-1)^2\, 2^2\, z_2\,(y\pm x)^{n-4} \ldots$$

$$+ (-1)^{n-2} .\; 2^{n-2}\; z_{n-2}\}.$$

Since $(y\pm x)^n - 2y\,(y\pm x)^{n-1} = (y\pm x)^{n-1}\,(-y\pm x)$

$$= (x^2 - y^2)\,(y\pm x)^{n-2},$$

the R. H. S. contains the factor $\{x^2 - y^2 + (-1)^2 . 2^2\, y_2\}$.

Hence provided y_2 is the quantity $\dfrac{y^2 - x^2}{4}$, R. H. S. = 0 for either sign and hence the result of elimination is the same in either cases, y_2 having the above value.

———

QUESTIONS FOR SOLUTION.

1238. (R. VYTHYANATHASWAMY):—If $f(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, find z_1, z_2, where

$$z_1 = p_1 x_1 y_1 + q_1 (x_1 y_2 + x_2 y_1) + r_1 x_2 y_2,$$

$$z_2 = p_2 x_1 y_1 + q_2 (x_1 y_2 + x_2 y_1) + r_2 x_2 y_2,$$

so that $f(z_1, z_2) = kf(x_1, x_2).\ f(y_1, y_2)$ *identically*, k being independent of x_1, x_2, y_1, y_2.

Conversely, if the bilinear forms z_1, z_2 are known, would it be possible to find a binary quadratic function f such that

$$f(z_1, z_2) \equiv kf(x_1, x_2), f(y_1, y_2)?$$

1239. (R. VYTHYANATHASWAMY):—If P, Q, R, S be four points in the Argand diagram, P′,Q′,R′,S′ their inverses *w. r. t.* a real or imaginary circle, shew that the cross ratios [PQRS] and [P′Q′R′S′] are conjugate complex numbers.

1240. (R. VYTHYANATHASWAMY):—S, S_1, S_2 are three conics of a four-point system, such that n-gons could be inscribed in S so as to be circumscribed to S_1 or S_2. A triangle is inscribed in S with two of its sides touching S_1, S_2 respectively. Shew that the envelope of the third side is a pair of conics each of which has n-gons inscribed in S, circumscribsd to itself.

1241. (A. C. L. WILKINSON):—Solve the differential equation,

$$2a^3 xy^2 \frac{d^3y}{dx^3} + 1 = 0.$$

(A special form of this equation, $a = \frac{2}{3}$, is given in Forsyth, *Differential Equations*, 3rd Edition, Miscellaneous Examples : 51 (ii), but in his Solutions, published in 1918, he states that he has not been able to obtain a solution in a finite form. A solution in finite form exists).

1242. (A. C. L. WILKINSON):—Solve the equation

$$\log \left(x \frac{dy}{dx} + y \right) + \frac{2y}{x\dfrac{dy}{dx} + y} = a.$$

1243. (A. C. L. WILKINSON):—Solve the differential equation

$$(y - px)(3y - px) = p, \text{ where } p = \frac{dy}{dx} \text{ as usual.}$$

Show that the p-discriminant is a cusp locus and that the envelope of the tangents at the cusps consists of the two hyperbolas $4\sqrt{3}\,xy = \pm 1$.

1244. (S. RAJANARAYANAN) :—Sum the series

$$(a-1)_0\, a_r - a_1 . a_{r-1} + (a+1)_2 . a_{r-2} - \dots (-1)^r (a+r-1)_r\, a_0$$

where n_r denotes the number of combinations of n things r at a time.

1245. (S. RAJANARAYANAN) :—Find the value of the infinite series

$$S_1 + \frac{S_2}{2\,!} + \frac{S_3}{3\,!} + \dots \dots$$

where S_r denotes the sum of the rth powers of the first n natural numbers.

1246. (S. RAJANARAYANAN) :—Find the value of the expression

$$\surd[a + \surd\{ab + \surd(ab^3 + \sqrt{ab^7 + \surd ab^{15} + \surd ab^{31} + \dots})\}]$$

1247. (N.B. MITRA) :—If $p\ [=(2n+1)]$ be a prime and if a, b, c b the digits in the units' place in

$$\frac{(2n\,!)+1}{p},\ \frac{(n\,!)\ +(-1)^n}{p},\ \frac{2^{4n}-1}{p},$$

respectively, when these are expressed in the scale of p, prove that

$$(a-b+c) \equiv 0 \pmod{p}.$$

1248. (A. NARASINGA RAO) :—If A sphere is moving about its centre so that the velocities of 3 points on it bear a constant ratio to one another, then the motion must be one of rotation about a fixed axis.

Prove this and make use of it to establish Bertrand's result that the curves for which the curvature is proportional to the torsion is a cylindrical helix.

1249. (K. C. SHAH, M.A.) :—Through a fixed point P in the plane of a $\triangle$ABC, a variable straight line is drawn cutting the sides BC, CA, AB in A′, B′, C′ respectively. If any point Q is taken on this straight line such that $\frac{1}{A'Q} + \frac{1}{B'Q} + \frac{1}{C'Q} = \frac{3}{PQ}$, prove that the locus of Q is a conic section.

Locate the position of P for the different species of the conic.

1250. (V. RAMASWAMI AIYAR) :—If five straight lines be tangents to a three-cusped hypocycloid, prove that the foci of the five parabolas touching the lines taken four at a time are all collinear.

1251. (C. KRISHNAMACHARY AND M. BHEEMASENA RAO) :—

Let $[A_r, A_n]$ stand for the persymmetric determinant,

$$\begin{vmatrix} A_r & A_{r+1} & A_{r+2} & \cdots & A_n \\ A_{+1} & A_{r+2} & A_{r+3} & \cdots & A_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & A_{n+1} & \cdots & \cdots & A_{2n-r} \end{vmatrix}$$

(*a*) If $A_s = 1 . 3 . 5 \ldots (2s-1)$, show that

$$[A_r, A_n] = A_r A_{r+1} \cdots A_n \cdot 2^{n-r} 4^{n-r-1} 6^{n-r-2} \ldots (2n-2r)^1.$$

(*b*) If $A_s = s!$,

$$[A_r A_n] = A_r A_{r+1} \cdots A_n \cdot 1!\, 2!\, 3! \ldots n-r!$$

(*c*) If $A_s = m (m+1) (m+2) \ldots (m+s-1)$,

$$[A_r A_n] = A_r A_{r+1} \cdots A_n \cdot 1!\, 2!\, 3! \ldots n-r!$$

(*d*) The value of the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ m+r & m+r+1 & \cdots & m+n \\ (m+r)_2, & (m+r+1)_2 & \cdots & (m+n)_2 \\ \cdots & \cdots & \cdots & \cdots \\ (m+r)_{n-r}, & (m+r+1)_{n-r} & \cdots & (m+n)_{n-r} \end{vmatrix}$$

is independent of m, where x_p denotes $x (x+1) (x+2) \ldots (x+p-1)$.

1252. (K. J. SANJANA, M.A.) :—Prove that for real integral values of x and y the equality $6x^2 + 2 = y^3$ is impossible, except in the single case when $x = 1$ and $y = 2$.

Examine if the equality holds for any other real rational values of x and y.

1253. (K. J. SANJANA, M.A. AND K. C. SHAH, M.A.) :—The centre of a conic described about the triangle of reference is at the point, whose trilinear co-ordinates are f, g, h : prove that the lengths (r_1, r_2) of its principal semi-axes are given by the equation—

$$4\Delta^2 r^4 - abc\, fgh\, r^2 \left\{ \frac{a^2}{\Delta - af} + \frac{b^2}{\Delta - bg} + \frac{c^2}{\Delta - ch} \right\} + \frac{a^2 b^2 c^2 f^2 g^2 h^2 \Delta}{(\Delta - af)(\Delta - bg)(\Delta - ch)} = 0,$$

where a, b, c, Δ have their usual meanings.